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SLICING AND INTERSECTION THEORY FOR CHAINS MODULO ν ASSOCIATED WITH REAL ANALYTIC VARIETIES

BY

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ABSTRACT. In a real analytic manifold a k dimensional (real) analytic chain is a locally finite sum of integral multiples of chains given by integration over certain k dimensional analytic submanifolds (or strata) of some k dimensional real analytic variety. In this paper, for any integer $\nu \geq 2$, the concepts and results of [6] on the continuity of slicing and the intersection theory for analytic chains are fully generalized to the modulo ν congruence classes of such chains.

1. **Introduction.** For a separable real analytic manifold M, a real analytic mapping $f\colon M \to \mathbf{R}^n$, and a k dimensional analytic chain T in M [4, 4.2.28] with $k \geq n$, it was shown in [6, 4.3] that the slice function $\langle T, f, \cdot \rangle$ is continuous on the set of points y in $\mathbf{R}^n$ for which

$$\dim (f^{-1}\{y\} \cap \operatorname{spt} T) \leq k - n \quad \text{and} \quad \dim (f^{-1}\{y\} \cap \operatorname{spt} \partial T) \leq k - n - 1.$$

Geometrically, for almost all y in $\mathbf{R}^n$, the slice $\langle T, f, y\rangle$ is the $k-n$ dimensional analytic chain in M given by oriented integration, counting multiplicities, along the fiber $f^{-1}\{y\} \cap \operatorname{spt} T$.

For any integer $\nu \geq 2$, two k dimensional analytic chains T_1 and T_2 are congruent modulo ν if there exists a third analytic chain Q such that $T_1 - T_2 = \nu Q$. The resulting congruence classes are called k dimensional analytic chains modulo ν in M. With such a congruence class S we associate the set $\operatorname{spt}^\nu S = \bigcap_{T \in S} \operatorname{spt} T$. Generalizing [6, 4.3], we prove in 4.1 our main result:

Slicing Modulo ν Theorem. *Suppose S is a k dimensional analytic chain modulo ν in M, f is an analytic map of M into $\mathbf{R}^n$, $k \geq n$, and*

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$\mathbf{R}^n \cap \{y\colon \dim(f^{-1}\{y\} \cap \mathrm{spt}^\nu S) \le k - n$ and $\dim(f^{-1}\{y\} \cap \mathrm{spt}^\nu \partial S) \le k - n - 1\}$. *Then there exists a continuous mapping from* Y *into the* $k-n$ *dimensional analytic chains modulo* ν *in* M *such that for every* $y \in Y$ *the value* $\langle S, f, y\rangle^\nu$, *which we call the* **slice modulo** ν **of** S **in** $f^{-1}\{y\}$, *satisfies the condition*:

If T *is a* k *dimensional analytic chain in* M *belonging to* S *with* $\dim(f^{-1}\{y\} \cap \mathrm{spt}\, T) \le k - n$ *and* $\dim(f^{-1}\{y\} \cap \mathrm{spt}\, \partial T) \le k - n - 1$, *then* $\langle T, f, y\rangle$ *belongs to* $\langle S, f, y\rangle^\nu$.

As an elementary example consider the four linear maps $g_1, g_2, g_3\colon \mathbf{R} \to \mathbf{R}^2$, $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $g_1(y) = (y, 0)$, $g_2(y) = (-y, 3^{1/2}y)$, $g_3(y) = (-y, -3^{1/2}y)$ for $y \in \mathbf{R}$ and $f(y, z) = y$ for $(y, z) \in \mathbf{R}^2$, the oriented half-line $H = \mathbf{E}^1 \llcorner \{y\colon y > 0\}$ in $\mathbf{R}$, and the one dimensional analytic chain

$$T = g_{1\#}H + g_{2\#}H + g_{3\#}H$$

in $\mathbf{R}^2$ (see Figure 1). We compute $\partial T = -3\delta_{(0,0)}$ and

$$\langle T, f, a\rangle = -\delta_{(a,-3^{1/2}a)} - \delta_{(a,3^{1/2}a)} \quad \text{for } a < 0,$$

$$\langle T, f, b\rangle = \delta_{(b,0)} \quad \text{for } b > 0 \quad \text{(see Figure 2).}$$

Hence $\langle T, f, \cdot\rangle$ is continuous on $\mathbf{R} \sim \{0\} = \mathbf{R} \sim f(\mathrm{spt}\, \partial T)$, in accord with [6, 4.3]; however,

$$\lim_{y\to 0-} \langle T, f, y\rangle = -2\delta_{(0,0)} \ne \delta_{(0,0)} = \lim_{y \to 0+} \langle T, f, y\rangle.$$

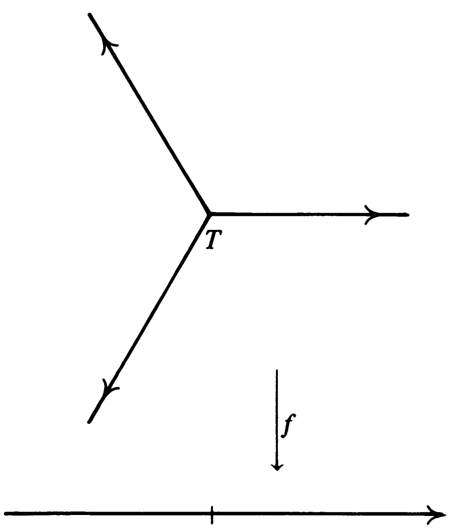

Figure 1

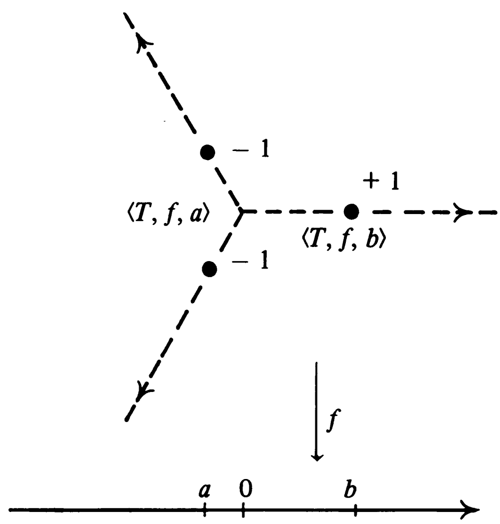

Figure 2

Nevertheless, letting S be the congruence class modulo 3 of T, we infer that $\operatorname{spt}^3 \partial S = \emptyset$, that $\langle S, f, 0\rangle^3$ is the congruence class of the two congruent chains $-2\delta_{(0,0)}, \delta_{(0,0)}$ and that the slice modulo 3, $\langle S, f, \cdot\rangle^3$, is continuous on all of $\mathbf{R}$, in accord with the slicing modulo ν theorem.

It follows from this theorem and the remarks in 4.4 that all of the results of [6, 4.3–5.11] carry over to the modulo ν case, including the resulting **intersection theory.**

The notation employed in the present paper is consistent with [6] and [4] (see the glossaries of [4, pp. 669–671]). As in [6] we let, for any two maps f: $A \to B$ and g: $A \to C$, the mapping $f \boxdot g$: $A \to B \times C$ be defined by $(f \boxdot g)(a) = (f(a), g(a))$ for $a \in A$.

Chains modulo ν arose classically in algebraic topology as members of various chain complexes having coefficient group $\mathbf{Z}_\nu = \mathbf{Z}/\nu\mathbf{Z}$ [8, III, 4(c)]. Rectifiable and flat chains modulo ν (especially $\nu = 2$) have been used in geometric measure theory in the study of multidimensional elliptic variation problems ([9], [5], [1], [2], [4, 5.3.21]). In [7] analytic chains and analytic chains modulo ν are used to define homology theories with coefficient groups $\mathbf{Z}$ and $\mathbf{Z}_\nu$ for the category of pairs of semianalytic sets and continuous maps between such pairs; questions concerning the extent of local orientability of real analytic varieties are treated there in these terms.

2. **Locally flat chains modulo ν.** The basic reference for flat chains modulo ν in $\mathbf{R}^n$ is [4, 4.2.26]. In this section we assume M is a separable Riemannian manifold and state appropriate generalizations of some of the results of [6, §3] and [4, 4.2.26] for *chains in M whose supports are not necessarily compact.*

Recalling [6, §3] and [4, p. 423] we endow the group $\mathcal{F}_k^{\mathrm{loc}}(M)$ of k dimensional locally integral flat chains in M with the **modulo ν topology** by associating with each pair (U, δ) such that

$$U \text{ is open}, \quad \operatorname{Clos} U \text{ is compact}, \quad \delta > 0,$$

a basic neighborhood of 0, $\mathbf{N}^\nu(U, \delta)$, consisting of those chains $T \in \mathcal{F}_k^{\mathrm{loc}}(M)$ for which there exist $Q \in \mathcal{R}_k(M)$, $R \in \mathcal{R}_k(M)$, and $S \in \mathcal{R}_{k+1}(M)$ with $\operatorname{spt}(T - R - \partial S - \nu Q) \subset M \sim U$, $\mathbf{M}(R) + \mathbf{M}(S) < \delta$.

The statements and proofs of [6, 3, 3.1, 3.2 (1)(3)(4)(5), 3.3, 3.4] readily generalize to the modulo ν case. Assuming $\{U_1, U_2, \cdots\}$ is a cover of M consisting of open sets with compact closures, we also see that the modulo ν topology is induced by the complete pseudometric dist^ν defined by

$$\operatorname{dist}^\nu(T_1, T_2) = \sum_{j=1}^{\infty} [\delta_j / 2^j(1 + \delta_j)]$$

where

$$\delta_j = \inf\{\delta\colon T_1 - T_2 \in \mathbf{N}^\nu(U_j, \delta)\}$$

for $T_1, T_2 \in \mathcal{F}_k^{\mathrm{loc}}(M)$; moreover, we say that T_1 and T_2 are **congruent modulo** ν, and write $T_1 \equiv T_2 \bmod \nu$, whenever $\mathrm{dist}^\nu(T_1, T_2) = 0$. The resulting congruence classes, which we call k dimensional **locally flat chains modulo** ν in M, are the cosets of the factor group

$$\mathcal{F}_k^{\mathrm{loc},\nu}(M) = \mathcal{F}_k^{\mathrm{loc}}(M)/\mathcal{F}_k^{\mathrm{loc}}(M) \cap \{T\colon \mathrm{dist}^\nu(T, 0) = 0\};$$

this group is complete in the induced metric. The operations on $\mathcal{F}_k^{\mathrm{loc}}(M)$ of restriction [6, 3.3], boundary (for $k \geq 1$), and $f_\#$ (see [6, 3.0(3)]) induce similar continuous operations on $\mathcal{F}_k^{\mathrm{loc},\nu}(M)$. For $T \in \mathcal{F}_k^{\mathrm{loc}}(M)$ we let $(T)^\nu \in \mathcal{F}_k^{\mathrm{loc},\nu}(M)$ be the coset of T. We also define the **modulo** ν **supports:**

$$\mathrm{spt}^\nu S = \bigcap_{R \in S} \mathrm{spt}\, R \quad \text{for } S \in \mathcal{F}_k^{\mathrm{loc},\nu}(M),$$

$$\mathrm{spt}^\nu T = \mathrm{spt}^\nu (T)^\nu \quad \text{for } T \in \mathcal{F}_k^{\mathrm{loc}}(M).$$

From the discussion of [4, p. 430] we infer that two **locally rectifiable currents** $T_1, T_2 \in \mathcal{R}_k^{\mathrm{loc}}(M)$ are congruent modulo ν if and only if there exists a third chain $Q \in \mathcal{R}_k^{\mathrm{loc}}(M)$ such that $T_1 - T_2 = \nu Q$; if, moreover, T_1 and T_2 are analytic chains in M [6, 4], then Q is also an analytic chain because

$$\mathrm{spt}\, Q \subset \mathrm{spt}\, T_1 \cup \mathrm{spt}\, T_2, \qquad \mathrm{spt}\, \partial Q \subset \mathrm{spt}\, \partial T_1 \cup \mathrm{spt}\, \partial T_2.$$

3. A constancy lemma and some consequences.

3.1. **Lemma** (compare [4, 4.1.31 (2)]). *Suppose B is a connected proper k dimensional class 1 submanifold of M. If $Q, R \in \mathcal{F}_k^{\mathrm{loc}}(M)$, $(\mathrm{spt}^\nu Q) \sim B$ and $(\mathrm{spt}^\nu R) \sim B$ are closed, and $(\mathrm{spt}^\nu \partial Q) \cup (\mathrm{spt}^\nu \partial R) \subset M \sim B$, then there exist integers i and j in $\{0, 1, \cdots, \nu - 1\}$ such that*

$$\Theta^k(\|Q\|, x) \equiv i \mod \nu \quad \textit{and} \quad \Theta^k(\|R\|, x) \equiv j \mod \nu \quad \textit{for all } x \in B,$$

$$\mathrm{spt}^\nu(jQ - iR) \subset M \sim B.$$

Proof. For any $x \in B$, there exist neighborhoods $U \subset \mathrm{Clos}\, U \subset V$ of x and a class 1 map p from V into $\mathbf{R}^k$ such that $V \cap \mathrm{spt}^\nu Q \subset B$, $V \cap \mathrm{spt}^\nu R \subset B$, $p \mid B \cap V$ is a proper class 1 diffeomorphism and $C = p(U \cap B)$ is an open cube in $\mathbf{R}^k$. Observing that $p_\#(Q \mid V) \llcorner C$ and $p_\#(R \mid V) \llcorner C$ extend [6, 3.3] to flat chains Q' and R' in $\mathbf{R}^k$ with

$$\begin{aligned} \mathrm{spt}^\nu \partial Q' &\subset [\mathrm{spt}^\nu \partial p_\#(Q \mid V)] \cup \mathrm{spt}\,(\partial[p_\#(Q \mid V) \llcorner C] - [\partial p_\#(Q \mid V) \llcorner C]) \\ &\subset p[V \cap \mathrm{spt}^\nu \partial Q] \cup \mathrm{Bdry}\, C = \mathrm{Bdry}\, C, \end{aligned}$$

$$\mathrm{spt}^\nu \partial R' \subset \mathrm{Bdry}\, C,$$

we infer from the modulo ν version of the second proposition of [4, 4.2.3] that

$$Q' \equiv i\mathbf{E}^k \llcorner C \mod \nu \quad \text{and} \quad R' \equiv j\mathbf{E}^k \llcorner C \mod \nu$$

for some i and j in $\{0, 1, \cdots, \nu - 1\}$; hence,

$$\Theta^k(\|Q\|, x) \equiv i \mod \nu \quad \text{and} \quad \Theta^k(\|R\|, x) \equiv j \mod \nu \quad \text{for all } x \in U \cap B,$$
$$U \cap \operatorname{spt}^\nu(jQ - iR) \subset p^{-1}[C \cap \operatorname{spt}^\nu(jQ' - iR')] = \emptyset$$

by the modulo ν analogue of the second corollary of [4, 4.1.15].

If follows that the sets

$$B_{i,j} = [B \sim \operatorname{spt}^\nu(jQ - iR)]$$
$$\cap \{x: \Theta^k(\|Q\|, x) \equiv i \bmod \nu \text{ and } \Theta^k(\|R\|, x) \equiv j \bmod \nu\}$$

for i and j in $\{0, 1, \cdots, \nu - 1\}$ form a relatively open disjointed cover of B; thus $B_{i,j} = B$ for some i, j.

3.2. **Proposition** (compare [3, 3.17], [6, 3.6]). *Suppose f is a locally Lipschizian map of M into an open subset N of $\mathbf{R}^n$, $T \in \mathcal{R}_k^{\mathrm{loc}}(M)$, and μ is a positive integer such that $f \mid \operatorname{spt}^\nu T$ is proper, $(\operatorname{spt}^\nu T) \sim \operatorname{spt}^\nu \partial T$ is locally connected, and $N \cap \{y: \operatorname{card}(f^{-1}\{y\} \cap \operatorname{spt}^\nu T) \leq \mu\}$ is dense in $N \sim f(\operatorname{spt}^\nu \partial T)$. Let*

$$Y = [N \sim f(\operatorname{spt}^\nu \partial T)] \cap \{y: \operatorname{card}(f^{-1}\{y\} \cap \operatorname{spt}^\nu T) < \infty\},$$

G be the class of all nonempty connected open subsets of $N \sim f(\operatorname{spt}^\nu \partial T)$ and, for $W \in G$, $\Gamma(W)$ be the set of all components of $f^{-1}(W) \cap \operatorname{spt}^\nu T$; also let

$$H = \bigcup \{\Gamma(W): W \in G\}, \qquad H^* = \{V \cap f^{-1}(Y): V \in H\}.$$

Then the following eight conclusions hold:

(1) *For each $V \in \Gamma(W)$ there exists an integer $\Delta(V) \in \{0, 1, \cdots, \nu - 1\}$ such that $f_\#(T \llcorner V) \equiv \Delta(V)\mathbf{E}^n \llcorner W \bmod \nu$.*

(2) *If $V \in \Gamma(W)$ and $\Delta(V) \neq 0$, then $f(V) = W$.*

(3) $\operatorname{Card}[\Gamma(W) \cap \{V: \Delta(V) \neq 0\}] \leq \mu$.

(4) *If $W \subset W'$ belong to G and $V' \in \Gamma(W')$, then $\Delta(V') \equiv \sum_{V' \supset V \in \Gamma(W)} \Delta(V)$ mod ν.*

(5) *The family H^* is a base for the relative topology of $f^{-1}(Y) \cap \operatorname{spt}^\nu T$.*

(6) *If $x \in f^{-1}(Y) \cap \operatorname{spt}^\nu T$, then $\Delta(V)$ has the same value, hereafter denoted $\Delta(x)$, for all sufficiently small neighborhoods V of x belonging to H.*

(7) *If $y \in Y$, then*

$$f_\#[T \llcorner f^{-1}(W)] \equiv \left[\sum_{x \in f^{-1}\{y\} \cap \operatorname{spt}^\nu T} \Delta(x)\right] \mathbf{E}^n \llcorner W \mod \nu$$

for all sufficiently small neighborhoods W of y belonging to G.

(8) *The function mapping $y \in Y$ onto $\sum_{x \in f^{-1}\{y\} \cap \mathrm{spt}^{\nu} T} \Delta(x)\delta_x$ is mass bounded by $\mu\nu$ and continuous in the modulo ν topology.*

Proof. To prove (1) we observe that

$$W \cap \mathrm{spt}^{\nu}\partial f_{\#}(T \llcorner V) \subset f[f^{-1}(W) \cap \mathrm{spt}^{\nu}\partial(T \llcorner V)]$$

$$\subset f([f^{-1}(W) \cap \mathrm{spt}^{\nu}\partial T] \cup [f^{-1}(W) \cap \mathrm{spt}^{\nu} T \cap \mathrm{Bdry}\ V]) = \emptyset,$$

and apply 3.1 with $B = W$, $Q = \mathbf{E}^n \llcorner W$, hence $i = 1$, and $R = f_{\#}(T \llcorner V)$ to choose $\Delta(V) \in \{0, 1, \cdots, \nu - 1\}$ such that

$$\mathrm{spt}^{\nu}[f_{\#}(T \llcorner V) - \Delta(V)\mathbf{E}^n \llcorner W] \subset \mathbf{R}^n \sim W,$$

$$f_{\#}(T \llcorner V) = f_{\#}(T \llcorner V) \llcorner W \equiv \Delta(V)\mathbf{E}^n \llcorner W \quad \mathrm{mod}\ \nu.$$

We verify (2) by noting that if $\Delta(V) \neq 0$, then

$$W \subset W \cap \mathrm{spt}^{\nu} f_{\#}(T \llcorner V) \subset f[f^{-1}(W) \cap \mathrm{spt}^{\nu}(T \llcorner V)]$$

$$\subset f[f^{-1}(W) \cap \mathrm{spt}^{\nu} T \cap \mathrm{Clos}\ V] = f(V).$$

To prove (3) we choose a point $y \in W$ with $\mathrm{card}\,(f^{-1}\{y\} \cap \mathrm{spt}^{\nu}\, T) \leq \mu$ and apply (2).

For the proof of (4) we have only to modify the proof of [3, 3.7 (4)] by changing the first and last equalities to congruences modulo ν.

(5) and (6) follow just as in the proofs of [3, 3.17 (5) (6)].

To prove (7) we abbreviate $F = f^{-1}\{y\} \cap \mathrm{spt}^{\nu}\, T$, choose open sets W_x and disjoint sets V_x for $x \in F$ so that $y \in W_x \in G$, $x \in V_x \in \Gamma(W_x)$, and $f_{\#}(T \llcorner V_x) \equiv \Delta(x)\mathbf{E}^n \llcorner W_x \bmod \nu$, and observe that F is finite,

$$\delta = \mathrm{dist}\left[y, \left(\bigcup_{x \in F} \mathrm{Bdry}\ W_x\right) \cup f\left(\mathrm{spt}^{\nu} T \sim \bigcup_{x \in F} V_x\right)\right]$$

is positive, and any $W \in G$ contained in $\mathbf{U}(y, \delta)$ satisfies

$$f^{-1}(W) \cap \mathrm{spt}^{\nu} T \subset \bigcup_{x \in F} V_x \quad \text{and}$$

$$f_{\#}[T \llcorner f^{-1}(W)] \equiv \sum_{x \in F} f_{\#}(T \llcorner [V_x \cap f^{-1}(W)])$$

$$\equiv \sum_{x \in F} \Delta(x)\mathbf{E}^n \llcorner (W_x \cap W) = \left(\sum_{x \in F} \Delta(x)\right)\mathbf{E}^n \llcorner W \quad \mathrm{mod}\ \nu.$$

The mass bound in (8) follows from (6) and (3). A proof of the modulo ν continuity results from modifying the wording of the proof of [6, 3.6 (9)] by changing spt T to $\mathrm{spt}^{\nu}\, T$ and the second equality in the computation of ∂S to a congruence modulo ν.

4. **Analytic chains modulo** ν. Assuming now that M is a separable m dimensional real analytic Riemannian manifold, we first recall the notion of **real analytic dimension** (which is defined in [6, 2.2]):

If A is an analytic subset of M, then $\dim A = \sup\{-1, k\colon A$ contains a nonempty k dimensional analytic submanifold of $M\}$.

If E is an arbitrary subset of M, then $\dim E = \inf\{k\colon E \subset \bigcup_{U \in \mathfrak{U}} A_U$ for some locally-finite open cover $\mathfrak{U}$ of M and k dimensional analytic subsets A_U of U for $U \in \mathfrak{U}\}$.

We will call S a k dimensional **analytic chain modulo** ν in M if and only if (compare [6, §4]) $S \in \mathcal{F}_k^{\mathrm{loc},\nu}(M)$, $\dim(\operatorname{spt}^\nu S) \le k$, $\dim(\operatorname{spt}^\nu \partial S) \le k-1$. It then follows that *there exists a k dimensional analytic chain T belonging to S with* $\operatorname{spt} T = \operatorname{spt}^\nu S$.

In fact, for every $x \in M$ there exist, by [4, 3.4.8 (11)], an open ball U about x and a finite family $\mathfrak{B}$ of disjoint k dimensional orientable analytic blocks B [6, §2] in $U \cap (\operatorname{spt}^\nu S \sim \operatorname{spt}^\nu \partial S)$ with orienting k vectorfields β_B such that the real analytic dimension of $Z = U \cap (\operatorname{spt}^\nu S \sim \operatorname{spt}^\nu \partial S \sim \bigcup \mathfrak{B})$ is less than k. Applying, for each $B \in \mathfrak{B}$, 3.1 to $Q = (\mathcal{H}^k \llcorner B) \wedge \beta_B$ and any locally flat chain R belonging to S (hence, $i = 1$, $\operatorname{spt}^\nu R = \operatorname{spt}^\nu S$, and $\operatorname{spt}^\nu \partial R = \operatorname{spt}^\nu \partial S$), we choose an integer $j_B \in \{1, 2, \cdots, \nu - 1\}$ so that

$$\operatorname{spt}^\nu[S - (j_B[\mathcal{H}^k \llcorner B] \wedge \beta_B)^\nu] = \operatorname{spt}^\nu[R - j_B(\mathcal{H}^k \llcorner B) \wedge \beta_B] \subset M \sim B.$$

Then

$$U \cap \operatorname{spt}^\nu \ S - \left[\left(\sum_{B \in \mathfrak{B}} j_B[\mathcal{H}^k \llcorner B] \wedge \beta_B\right)^\nu\right]$$

is contained in $Z \cup \operatorname{spt}^\nu \partial S$, has $\mathcal{H}^k$ measure zero by [6, 2.2 (5) (4)], and is empty by [4, 4.2.26 (4.2.14)$^\nu$]. Moreover

$$U \cap \operatorname{spt}^\nu S = U \cap \operatorname{Clos} \bigcup \mathfrak{B} = U \cap \operatorname{spt} \sum_{B \in \mathfrak{B}} j_B(\mathcal{H}^k \llcorner B) \wedge \beta_B.$$

To construct T globally we choose open balls $U_1, U_2, \cdots$ along with analytic chains $T_1, T_2, \cdots$ in M such that $\{U_1, U_2, \cdots\}$ is a locally finite open cover of M and

$$U_i \cap \operatorname{spt}^\nu[S - (T_i)^\nu] = \varnothing, \qquad U_i \cap \operatorname{spt}^\nu S = U_i \cap \operatorname{spt} T_i.$$

Then we use [6, 2.2 (7)] to select, for $i \in \{1, 2, \cdots\}$, open balls V_i with closure in U_i such that $M \subset \bigcup_{i=1}^\infty V_i$ and

$$\mathcal{H}^k([(\operatorname{spt}^\nu S) \cup (\operatorname{spt} T_i)] \cap \operatorname{Bdry} V_j) = 0$$

for i in $\{0, 1, \cdots\}$ and j in $\{0, 1, \cdots, \nu - 1\}$, define the analytic chain

$$T = T_1 \llcorner V_1 + \sum_{i=2}^{\infty} T_i \llcorner \left(V_i \sim \bigcup_{j=1}^{i-1} V_j\right),$$

and conclude that $\operatorname{spt} T = \operatorname{spt}^\nu T$ and that $\mathcal{H}^k(\operatorname{spt}^\nu[S - (T)^\nu]) = 0$; hence $S = (T)^\nu$ by [4, 4.2.26 $(4.2.14)^\nu$].

4.1. **Proof of the Slicing Modulo ν Theorem.** We fix an analytic chain T belonging to S with $\operatorname{spt} T = \operatorname{spt}^\nu S$; let

$$W = \mathbf{R}^n \cap \{w : \dim(f^{-1}\{w\} \cap \operatorname{spt} T) \leq k - n \text{ and } \dim(f^{-1}\{w\} \cap \operatorname{spt} \partial T) \leq k - n - 1\},$$

hence $W \subset Y$, and observe that it suffices to prove the following statement:

There exists a continuous map $\mathcal{S}$ *from* Y *into the* $k - n$ *dimensional analytic chains modulo* ν *in* M *such that* $\mathcal{S}(w) = (\langle T, f, w\rangle)^\nu$ *for every* $w \in W$.

In fact suppose this statement is true and T' is another analytic chain belonging to S with

$$W' = \mathbf{R}^n \cap \{w : \dim(f^{-1}\{w\} \cap \operatorname{spt} T') \leq k - n \text{ and}$$
$$\dim(f^{-1}\{w\} \cap \operatorname{spt} \partial T') \leq k - n - 1\}.$$

Then $T' - T = \nu Q$ for some k dimensional analytic chain Q, and for any $w \in W' \cap W$,

$$\dim(f^{-1}\{w\} \cap \operatorname{spt} Q) \leq k - n, \qquad \dim(f^{-1}\{w\} \cap \operatorname{spt} \partial Q) \leq k - n - 1;$$

hence, by [6, 4.3], $\langle Q, f, w\rangle$ is an analytic chain in M, and

$$\langle T', f, w\rangle - \langle T, f, w\rangle = \nu\langle Q, f, w\rangle \equiv 0 \mod \nu.$$

The two continuous maps $\mathcal{S} \mid W'$ and $(\langle T', f, \cdot\rangle)^\nu$ of W' agree on the dense [6, 2.2 (7)] subset $W' \cap W$, hence are equal.

To prove the statement we consider three cases.

Case 1. *M is an open subset of* $\mathbf{R}^m$, *$\operatorname{spt} T$ is compact, and $k = n$.* Here we use 3.1 and [4, 3.4.8 (11)], reasoning as in [4, 4.2.28] to see that $\operatorname{spt}^\nu T \sim \operatorname{spt}^\nu \partial T$ is locally connected, note that $f \mid \operatorname{spt} T$ is proper because $\operatorname{spt} T$ is compact, choose, according to [6, 2.11 (1)] a positive integer μ so that the set

$$\mathbf{R}^n \cap \{y : \operatorname{card}(f^{-1}\{y\} \cap \operatorname{spt} T) > \mu\}$$

has Lebesgue measure zero, and then apply 3.2 (7) by setting

$$\mathcal{S}(y) = \left(\sum_{x \in f^{-1}\{y\} \cap \operatorname{spt}^\nu T} \Delta(x)\delta_x\right)^\nu \quad \text{for } y \in Y.$$

The statement follows by comparing 3.2 (8) (6) (1) and [6, 3.6 (8) (6) (1)].

Case 2. *M is an open subset of* $\mathbf{R}^m$, spt T *is compact, and* $k > n$. Here we assume, for contradiction, that the statement is false. Since W is dense in $\mathbf{R}^n$ by [6, 2.2 (7)], there then must exist a point $y \in Y$ such that $(\langle T, f, w\rangle)^\nu$ fails to converge as w approaches y in W. Assured by [4, 4.2.17] and [6, 4.2] that the family $\{\langle T, f, w\rangle : w \in W\}$ is relatively compact in $\mathbf{I}_k(M)$ we choose two sequences $w_{1,1}, w_{1,2}, w_{1,3} \cdots$ and $w_{2,1}, w_{2,2}, w_{2,3} \cdots$ in W converging to y and integral currents L_1 and L_2 in $\mathbf{I}_k(M)$ such that $\operatorname{spt}^\nu(L_1 - L_2)$ is nonempty and $\langle T, f, w_{i,j}\rangle$ approaches L_i, for $i \in \{1, 2\}$, as j approaches ∞. Since

$$\operatorname{spt}^\nu L_i \subset f^{-1}\{y\} \cap \operatorname{spt}^\nu S, \qquad \operatorname{spt}^\nu \partial L_i \subset f^{-1}\{y\} \cap \operatorname{spt}^\nu \partial S,$$

$(L_i)^\nu$ is an analytic chain modulo ν in M for $i \in \{1, 2\}$. Thus we may select an open set U in $M \sim \operatorname{spt}^\nu \partial(L_1 - L_2)$ and an orthogonal projection $p: M \to \mathbf{R}^{k-n}$ such that $B = U \cap \operatorname{spt}^\nu(L_1 - L_2)$ is a nonempty connected $k - n$ dimensional analytic submanifold of M and $p \mid B$ is an analytic isomorphism. Fixing, by [6, 2.2 (7)], $z \in p(B)$ so that

$$\dim (f^{-1}\{y\} \cap p^{-1}\{z\} \cap \operatorname{spt}^\nu S) = 0, \qquad f^{-1}\{y\} \cap p^{-1}\{z\} \cap \operatorname{spt}^\nu \partial S = \varnothing$$

we infer from 3.1, 3.2 (6), and Case 1 that

$$\langle (L_1 - L_2)^\nu, p, z\rangle^\nu \mid U \neq (0)^\nu .$$

Next using [6, 3.2] we pass to subsequences, without changing notations, so that there exist for $i \in \{1, 2\}$ and $j \in \{1, 2, \cdots\}$ rectifiable chains $R_{i,j} \in \mathfrak{R}_{k-n}(M)$, $S_{i,j} \in \mathfrak{R}_{k-n+1}(M)$ with

$$\operatorname{spt}(L_i - \langle T, f, w_{i,j}\rangle - R_{i,j} - \partial S_{i,j}) \subset M \sim U,$$

$$\mathbf{M}(R_{i,j}) + \mathbf{M}(S_{i,j}) \leq j^{-1}\mathfrak{L}^{k-n}\mathbf{U}(z, j^{-1}),$$

and then apply [6, 2.2 (7)] and [4, 4.3.6, 4.3.2 (2)] to choose points $z_{i,j} \in \mathbf{R}^{k-n} \cap \mathbf{U}(z, j^{-1})$ such that

$$\dim (p^{-1}\{z_{i,j}\} \cap [(f^{-1}\{w_{i,j}\} \cap \operatorname{spt} T) \cup \operatorname{spt} L_i]) \leq 0,$$

$$p^{-1}\{z_{i,j}\} \cap [(f^{-1}\{w_{i,j}\} \cap \operatorname{spt} \partial T) \cup \operatorname{spt} \partial L_i] = \varnothing,$$

$$\langle R_{i,j}, p, z_{i,j}\rangle \in \mathfrak{R}_0(M), \qquad \langle S_{i,j}, p, z_{i,j}\rangle \in \mathfrak{R}_1(M),$$

$$\mathbf{M}\langle R_{i,j}, p, z_{i,j}\rangle + \mathbf{M}\langle S_{i,j}, p, z_{i,j}\rangle \leq j^{-1},$$

and deduce from [6, 3.5 (2)] that the slices

$$\langle L_i - \langle T, f, w_{i,j}\rangle, p, z_{i,j}\rangle \mid U = [\langle R_{i,j}, p, z_{i,j}\rangle + (-1)^{k-n}\partial\langle S_{i,j}, p, z_{i,j}\rangle] \mid U$$

approach 0 as j approaches ∞ for $i \in \{1, 2\}$. Finally applying Case 1 twice and [6, 4.5], we obtain the desired contradiction by computing

$$(0)^\nu \neq \langle (L_1 - L_2)^\nu, p, z\rangle^\nu \mid U = \lim_{j\to\infty} (\langle L_1, p, z_{1,j}\rangle - \langle L_2, p, z_{2,j}\rangle)^\nu \mid U$$

$$= \lim_{j\to\infty} (\langle\langle T, f, w_{1,j}\rangle, p, z_{1,j}\rangle - \langle\langle T, f, (w_{2,j}\rangle, p, z_{2,j}\rangle)^\nu | U$$

$$= \lim_{j\to\infty} (\langle T, f \boxdot p, (w_{1,j}, z_{1,j})\rangle - \langle T, f \boxdot p, (w_{2,j}, z_{2,j})\rangle)^\nu | U$$

$$= (\langle (T)^\nu, f \boxdot p, (y, z)\rangle^\nu - \langle (T)^\nu, f \boxdot p, (y, z)\rangle^\nu) \mid U.$$

Case 3. *General case*. Assuming the theorem false, we choose, by the modulo ν analogue of [6, 3.2 (1)] an analytic isomorphism ϕ of some open subset V of M onto $\mathbf{R}^m \cap \mathbf{U}(0, 2)$, $U = \phi^{-1}[\mathbf{U}(0, 1)]$, a countable subset C of W, and a point $y \in Y \cap \operatorname{Clos} C$ such that $\langle T, f, w\rangle^\nu \mid U = \langle T \mid U, f \mid U, w\rangle^\nu$ fails to converge as w approaches y in C. Choosing, by [6, 2.2 (7)], r between 1 and 2 such that

$$\dim (|\phi|^{-1}\{r\} \cap \operatorname{spt} T) \leq k - 1,$$
$$\dim (|\phi|^{-1}\{r\} \cap f^{-1}\{y\} \cap \operatorname{spt} T) \leq k - n - 1,$$
$$\dim (|\phi|^{-1}\{r\} \cap f^{-1}\{w\} \cap \operatorname{spt} T) \leq k - n - 1,$$

we infer that

$$\langle \phi_\#(T|V) \llcorner \mathbf{U}(0, r), f \circ \phi^{-1}, w\rangle^\nu \mid \mathbf{U}(0, 1) = (\phi|U)_\# \langle T|U, f|U, w\rangle^\nu$$

fails to converge as w approaches y in C, which contradicts either Case 1 or Case 2 with S, T, f replaced by $[\phi_\#(T \mid V) \llcorner \mathbf{U}(0, r)]^\nu$, $\phi_\#(T \mid V) \llcorner \mathbf{U}(0, r)$, $f \circ \phi^{-1}$.

4.2. **Remark.** With T, f, Y, W as in 4.1, we see that the equation $(\langle T, f, y\rangle)^\nu = \langle (T)^\nu, f, y\rangle^\nu$ holds whenever $y \in W$. For $y \in Y \sim W$ however, this may fail to be true even though the slice $\langle T, f, y\rangle$, as defined in [4, 4.3], is an analytic chain in M. For example if $f\colon \mathbf{R} \to \mathbf{R}$ is the identity map and

$$T = \mathbf{E}^1 \llcorner \{x : x > 0\} - \mathbf{E}^1 \llcorner \{x : x < 0\},$$

then

$$(\langle T, f, 0\rangle)^2 = (0)^2 \neq (\delta_0)^2 = \langle (T)^2, f, 0\rangle^2.$$

4.3. For S, f, Y as in the Slicing Modulo ν Theorem and $y \in Y$, we readily infer, by continuity and the definition of the slice [4, 4.3.1], the following four statements:

(1) $\operatorname{spt}^\nu \langle S, f, y\rangle^\nu \subset f^{-1}\{y\} \cap \operatorname{spt}^\nu S$.

(2) $\partial\langle S, f, y\rangle^\nu = (-1)^n \langle \partial S, f, y\rangle^\nu$ in case $k > n$.

(3) $\langle S, f, y\rangle^\nu \mid U = \langle S \mid U, f \mid U, y\rangle^\nu$ whenever U is an open subset of M.

(4) $\langle S, h \circ f, h(y)\rangle^\nu = \langle S, f, y\rangle^\nu$ whenver h is an orientation-preserving analytic isomorphism of $\mathbf{R}^n$.

4.4. We will now discuss how *all of the statements and most of the proofs of* [6, 4.4–5.11] *carry over to the modulo* ν *case*. The wording of these propositions should be modified by replacing "chain, spt, $\mathcal{F}_*^{\mathrm{loc}}(M)$, and $\langle\ ,\ ,\ \rangle$" to "chain modulo ν, spt^ν, $\mathcal{F}_*^{\mathrm{loc},\nu}$, and $\langle\ ,\ ,\ \rangle^\nu$". We will number such generalizations by using the superscript ν above the number of the corresponding proposition of [6].

$(4.4)^\nu$ Here it is necessary to define $\langle f_\# T, g, y\rangle^\nu$ as $f_\#(\langle T, g \circ f, y\rangle^\nu)$ and then observe by 4.2 and [6, 4.4] that this is consistent in case $f_\# T$ is an analytic chain modulo ν.

$(4.5)^\nu$ requires a new proof; we prove the first conclusion by considering two cases:

Case 1. $s = n + l$. Here, assuming the theorem false, we choose a point

$$x \in \operatorname{spt}^\nu [\langle S, f \boxdot g, (a, b)\rangle^\nu - \langle\langle S, f, a\rangle^\nu, g, b\rangle^\nu],$$

then an open ball U about x in M and i and j in $\{0, 1, \cdots, \nu - 1\}$ so that

$$U \cap \operatorname{spt}^\nu \partial S = \emptyset, \qquad (\operatorname{Clos} U) \cap f^{-1}\{a\} \cap g^{-1}\{b\} \cap \operatorname{spt}^\nu S = \{x\},$$

$$\langle S, f \boxdot g, (a, b)\rangle^\nu \mid U = i(\boldsymbol{\delta}_x)^\nu \mid U \neq j(\boldsymbol{\delta}_x)^\nu \mid U = \langle\langle S, f, a\rangle^\nu, g, b\rangle^\nu \mid U.$$

Since

$$\begin{aligned} 0 < \delta &= \inf\{|(a, b) - (f \boxdot g)(w)| : w \in (\operatorname{Bdry} U) \cap \operatorname{spt}^\nu S\} \\ &\le \inf\{|b - g(w)| : w \in (\operatorname{Bdry} U) \cap \operatorname{spt}^\nu \langle S, f, a\rangle^\nu\}, \end{aligned}$$

$V = U \cap (f \boxdot g)^{-1}(\mathbf{U}[(a, b), \delta])$ is open and the two functions, $(f \mid V) \boxdot (g \mid V) \mid \operatorname{spt}^\nu S$ and $(g \mid V) \mid \operatorname{spt}^\nu \langle S, f, a\rangle^\nu$ are proper maps. Applying 3.2 (7) twice–with M, N, f, and T replaced:

first, by V, $\mathbf{U}[(a, b), \delta]$, $(f \mid V) \boxdot (g \mid V)$, and any analytic chain belonging to $S \mid V$, and

second, by V, $\mathbf{U}(b, \delta)$, $g \mid V$, and any analytic chain belonging to $\langle S, f, a\rangle^\nu \mid V$, we choose open balls, W about a in $\mathbf{R}^n$ and Z about b in $\mathbf{R}^l$, and

$$\alpha = f \mid U \cap (f \boxdot g)^{-1}(W \times Z), \qquad \beta = g \mid U \cap (f \boxdot g)^{-1}(W \times Z),$$
$$R = S \mid U \cap (f \boxdot g)^{-1}(W \times Z)$$

such that $W \times Z \subset \mathbf{U}[(a, b), \delta]$ and

$$(\alpha \boxdot \beta)_\# R = i(\mathbf{E}^n \times \mathbf{E}^l)^\nu \mid (W \times Z), \qquad \beta_\# \langle R, \alpha, a\rangle^\nu = j(\mathbf{E}^l)^\nu \mid Z.$$

Letting $q\colon W \times Z \to W$, $h\colon U \cap (f \boxdot g)^{-1}(W \times Z) \to W \times Z$ be given by $q(w, z) = w$, $h(u) = (a, \beta(u))$ for $(w, z) \in W \times Z$ and $u \in U \cap (f \boxdot g)^{-1}(W \times Z)$, we use $(4.4)^\nu$ and the modulo ν version of [4, 4.1.15] to compute

$$i(\delta_a \times \mathbf{E}^l)^\nu \mid (W \times Z) = \langle(\alpha \boxdot \beta)_\# R, q, a\rangle^\nu = (\alpha \boxdot \beta)_\# \langle R, \alpha, a\rangle^\nu = h_\# \langle R, \alpha, a\rangle^\nu$$
$$= [(\delta_a)^\nu \mid W] \times \beta_\# \langle R, \alpha, a\rangle^\nu = j(\delta_a \times \mathbf{E}^l)^\nu \mid (W \times Z),$$

hence $i = j$, a contradiction.

Case 2. $s > n + l$. Again assuming the theorem is false, we choose an open set U in $M \sim \operatorname{spt}^\nu \partial S$ and an analytic map $p\colon U \to \mathbf{R}^{s-n-l}$ such that

$$B = U \cap \operatorname{spt}^\nu(\langle S, f \boxdot g, (a, b)\rangle^\nu - \langle\langle S, f, a\rangle^\nu, g, b\rangle^\nu)$$

is a nonempty connected analytic submanifold of M and $p \mid B$ is an analytic isomorphism, let $w \in p(B)$, abbreviate $\overline{S} = S \mid U$, $\overline{f} = f \mid U$, $\overline{g} = g \mid U$ and use 3.1, 3.2 (6) (1), Case 1, and 4.3 (4) to derive the contradiction

$$(0)^\nu \neq \langle\langle \overline{S}, \overline{f} \boxdot \overline{g}, (a, b)\rangle^\nu - \langle\langle \overline{S}, \overline{f}, a\rangle^\nu, \overline{g}, b\rangle, p, w\rangle^\nu$$
$$= \langle\langle \overline{S}, \overline{f} \boxdot \overline{g}, (a, b)\rangle^\nu, p, w\rangle^\nu - \langle\langle \overline{S}, \overline{f}, a\rangle^\nu, \overline{g} \boxdot p, (b, w)\rangle^\nu$$
$$= \langle \overline{S}, (\overline{f} \boxdot \overline{g}) \boxdot p, ((a, b), w)\rangle^\nu - \langle \overline{S}, \overline{f} \boxdot (\overline{g} \boxdot p), (a, (b, w))\rangle^\nu = (0)^\nu.$$

The second conclusion of $(4.5)^\nu$ is readily obtained from the first as in [6, 4.5].

$(4.6)^\nu$ through $(4.9)^\nu$ now follows as in [6, 4.6–4.9].

In $(\S 5)^\nu$, M and N need only be orientable modulo ν with orienting modulo ν cycles $\mathfrak{M}$, $\mathfrak{N}$. The **intersection modulo** ν **of** Q **and** R, $Q \cap^\nu R \in \mathcal{F}_{q+r-m}^{\mathrm{loc},\nu}(M)$, is to be defined in a manner similar to [6, 5.0 (1)] where h now need only preserve orientation modulo ν; also we make the obvious generalization to obtain the notion that $\{S, T\}$ **intersect suitably modulo** ν.

For $(5.1)^\nu$ we assume now that Q and R are *analytic chains modulo* ν satisfying

$$\dim(b^{-1}\{y\} \cap \operatorname{spt}^\nu Q) \le q - k, \qquad \dim(b^{-1}\{y\} \cap \operatorname{spt}^\nu \partial Q) \le q - k - 1,$$
$$\dim(c^{-1}\{z\} \cap \operatorname{spt}^\nu R) \le r - l, \qquad \dim(c^{-1}\{z\} \cap \operatorname{spt}^\nu \partial R) \le r - l - 1,$$

and obtain the desired equations by continuity from [6, 5.1].

$(5.2)^\nu$ through $(5.6)^\nu$ may now be deduced as in [6, 5.2–5.6].

In $(5.7)^\nu$, L is an analytic chain modulo ν, $\mathfrak{M} = (\mathbf{E}^m \mid M)^\nu$, and $\mathfrak{N} = (\mathbf{E}^n \mid N)^\nu$, and we apply [6, 5.7] to any member T of L to infer that

$$\tau_{\#}\langle \mathfrak{M} \times L, f \circ \sigma, 0\rangle^\nu = (\tau_{\#}\langle (\mathbf{E}^m \mid M) \times T, f \circ \sigma, 0\rangle)^\nu = (T)^\nu = L$$
$$= (-1)^{(l+n)n} (\tilde{\sigma}_{\#} \langle T \times (\mathbf{E}^n \mid N), \tilde{f} \circ \tilde{\tau}, 0\rangle)^\nu$$
$$= (-1)^{(l+n)n} \tilde{\sigma}_{\#} \langle L \times \mathfrak{N}, \tilde{f} \circ \tilde{\tau}, 0\rangle^\nu.$$

$(5.8)^\nu$ through $(5.11)^\nu$, Case 4, follow from the proofs of [6, 5.8–5.10, Case 4] by use of 3.2 in place of [4, 4.1.31] and [4, 4.2.28 $(4.2.14)^\nu$] in place of [4, 4.1. 20].

To modify the proof of Case 5 it suffices to use 3.2 (7) to choose ρ and σ small enough so that

$$(f \mid V)_{\#}[(S \times T) \mid V] = (-1)^{(m-s)t} i_x [\mathbf{E}^m \mid \mathbf{U}(0, \sigma)]^\nu,$$
$$\langle (S \times T) \mid V, f \mid V, 0\rangle^\nu = (-1)^{(m-s)t} i_x (\delta_x \mid V)^\nu.$$

For Case 6, instead of using [6, 4.1], we observe that

If R is a k dimensional analytic chain modulo ν in an open subset M of $\mathbf{R}^m$ and $\langle R, \mathbf{p}_\lambda \mid M, y\rangle^\nu$ is zero for all $\lambda \in \Lambda(m, k)$ and $\mathfrak{L}^k$ almost all y in $\mathbf{R}^k$, then R equals zero.

In fact, otherwise for any regular point x of $\operatorname{spt}^\nu R \sim \operatorname{spt}^\nu \partial R$ there is a $\lambda \in \Lambda(m, k)$ such that $\dim \mathbf{p}_\lambda[\operatorname{Tan}(\operatorname{spt}^\nu R, x)] = k$, and, by [4, 3.1.18], an open neighborhood U of x in $M \sim \operatorname{spt}^\nu \partial R$ such that $B = U \cap \operatorname{spt}^\nu R$ is a connected analytic submanifold of M and $\mathbf{p}_\lambda \mid B$ is an analytic isomorphism. Then by 3.1, $(\mathbf{p}_\lambda \mid U)_{\#}(R \mid U)^\nu$ equals $[j\mathbf{E}^k \mid \mathbf{p}_\lambda(B)]^\nu$ for some $j \in \{1, 2, \cdots, \nu - 1\}$; hence, by 3.2,

$$\langle R, \mathbf{p}_\lambda \mid M, \mathbf{p}_\lambda(x)\rangle^\nu \mid U = \langle R \mid U, \mathbf{p}_\lambda \mid U, \mathbf{p}_\lambda(x)\rangle^\nu = (j\delta_x \mid U)^\nu \neq (0)^\nu$$

whenver $x \in B$, a contradiction.

Finally Case 7 follows from Case 5, Case 6, and $(5.11)^\nu$ (3) (4) as before.

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